ELECTROSTATICS

In electrostatics we deal with the behaviour of charges at rest. While studying electrostatics we often talk about two kinds of charges; **source charges** and **test charges**. If a charge is placed in the vicinity of another charge and experiences Coulomb's force, we call that charge as the **test charge** and the one in whose vicinity the test charge is placed is called **source charge.** The study of electrostatics requires the source charge to be stationary (though the test charge may be moving).

Source charge as the name suggests is the source of something- in this case is the source of electric force field. We call this the **electric field.** Electric field is the region around charge in which if a test charge is placed it experiences an electric force. The electric force is the **Coulomb's force**. The Coulomb's force is given by the expression

$$F = \frac{1}{4\pi\epsilon_0}\frac{qQ}{r^2}\hat{r} \tag{1}$$

This gives the force on test charge Q due to the source charge q which is at rest at a distance r away from Q. The constant ϵ_0 is called the permittivity of free space whose value in SI units is

$$\epsilon_0 = 8.85 \times 10^{-12} C^2 N^{-1} m^2$$

The expression for electric force can be written as

$$F = QE$$

Where

$$E = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r} \tag{2}$$

is called the **electric field strength.** That is the electric field strength is the Coulomb's force per unit test charge due to the source charge q. Electric field strength is a vector quantity.

Electric flux

Graphically we represent the electric field strength by arrows called field lines. *The number of field lines passing through a given surface in the direction perpendicular to the area of surface is called the flux of E through that surface.* Mathematically it is given by

$$d\phi = E \cdot da$$

Where da is the area vector in the normal direction to the element. The total flux through a finite area is given by

$$\phi = \iint E \cdot da$$

If the surface is closed we write

$$\phi = \oiint E \cdot da$$

Gauss Law:

It states that the total electric flux through a closed surface is equal to $\frac{1}{\epsilon_0}$ *times the total charge enclosed by the surface when the medium is vacuum.*

Mathematically if q is the charge enclosed by a surface

$$\phi = \oiint E \cdot da = \frac{q}{\epsilon_0} \tag{3}$$

The proof of this law comes from the definition of electric flux itself.

Consider a closed surface S surrounding charge +q (fig 1).

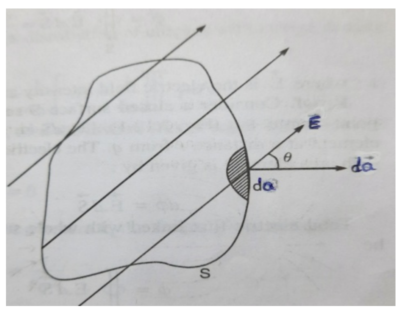

Figure 1

The total electric flux through this closed surface is

$$\phi = \oiint E \cdot da$$

Using $E = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$, we get

$$\phi = \oiint \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r} \cdot da = \frac{q}{4\pi\epsilon_0}\oiint \frac{\hat{r} \cdot da}{r^2} = \frac{q}{4\pi\epsilon_0}\oiint d\Omega$$

Where $d\Omega = \frac{\hat{r} \cdot da}{r^2}$ is called the solid angle and its value for the closed surface is 4π. That is $\oiint d\Omega = 4\pi$.

$$\therefore \phi = \frac{q}{4\pi\epsilon_0} 4\pi = \frac{q}{\epsilon_0}$$

This is Gauss law.

Now if instead of a single charge +q we have number of charges then the total charge enclosed inside will be sum of all the charges.

Therefore $Q_{enc.} = \sum_{i=1}^{n} q_i$.

Then $\phi = \frac{Q_{enc.}}{\epsilon_0}$

From Gauss law it is clear that it is the total charge inside the closed surface which contributes to the electric flux. If the charge is outside of the surface the total flux is zero, because field lines passing through one side and out through the other.

Gauss law discussed above has an integral form and hence it is called the **integral form** of Gauss law. Now we will turn this into differential form.

By applying the Gauss divergence theorem for vector E

$$\oiint E \cdot da = \iiint (\nabla \cdot E) d\tau \tag{4}$$

If ρ is the volume charge density we can write total charge enclosed as

$$Q_{enc.} = \iiint \rho d\tau. \tag{5}$$

So the integral form of Gauss law using equation (4) and (5) becomes

$$\iiint (\nabla \cdot E) d\tau = \iiint \frac{\rho}{\epsilon_0} d\tau$$

Since it holds for any volume the integrands must be equal.

$$\nabla \cdot E = \frac{\rho}{\epsilon_0} \tag{6}$$

This is the Gauss law in **differential form.** This gives us the divergence of E and is the first Maxwell equation of electrodynamics

Line integral of electric field

Suppose there is a positive charge +q (source charge) somewhere in space producing an electric field E. A test charge Q brought in this field will experience an electric force. To carry the charge Q towards the source charge against the electric forces a force F is required which can be given as

$$F = -QE \tag{7}$$

(negative sign indicates that charge Q is carried against the electric force)

Let the test charge is displaced through infinitesimal distance dl, then the small work done in making this displacement is

$$dW = F \cdot dl = -QE \cdot dl$$

If the test charge is moved through an arbitrary path from points a to b, total work done in doing so is

$$W = \int dW = -Q \int_a^b E \cdot dl$$

Now if the work is done in carrying one unit of charge i.e. $Q = 0$ we will have

$$W = -\int_a^b E \cdot dl \tag{8}$$

The term $\int_a^b E \cdot dl$ is called the line integral of electric field along a path between two points a and b and is defined as the work done in moving a unit positive charge along that path. The negative sign is a matter of convention as it is carried from the equation $F = -QE$.

Conservative nature of electric field: the curl of E

From the vector calculus we have studied that conservative field is one whose curl is zero. To show that electric field is conservative field we need to prove that curl of E is zero.

The electric field for a point charge q at origin is

$$E = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$$

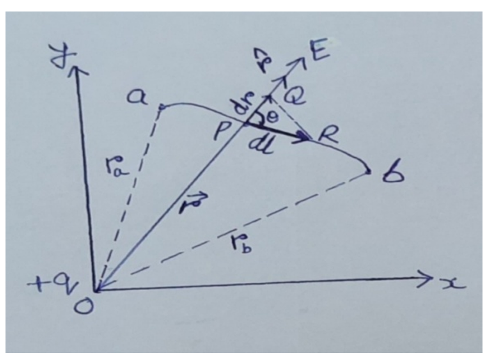

Figure 2

The line integral of electric field between two points a and b is

$$\int_a^b E \cdot dl$$

Now $E \cdot dl = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r} \cdot dl = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}dr$ $[\hat{r} \cdot dl = dr]$[1]

Therefore,

$$\int_a^b E \cdot dl = \frac{q}{4\pi\epsilon_0}\int_a^b \frac{1}{r^2} = \frac{-q}{4\pi\epsilon_0}\frac{1}{r}\bigg|_{r_a}^{r_b} = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r_a} - \frac{1}{r_b}\right)$$

$$\int_a^b E \cdot dl = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r_a} - \frac{1}{r_b}\right) \tag{9}$$

Where r_a and r_b are the distances of point a and b from the origin. From this equation it is clear that the line integral of electric field is not path dependent but depends only on the end points of the path along which the integral is taken. If the integral is taken around a closed path points a and b would coincide and the integral would then be zero (because then $r_a = r_b$).

$$\oint E \cdot dl = 0 \tag{10}$$

According to the stokes theorem

$$\oint E \cdot dl = \iint (\nabla \times E) \cdot da \tag{11}$$

Using (11) in (10) we therefore have

$$\nabla \times E = 0 \tag{12}$$

This result proves that E is a conservative field. Note that this result can also be understood from the electric field lines. Electric field lines either diverge from a positive charge or converge at a negative charge. These lines do not form circulations and hence are curl free. Thus the electric field represented by these field lines has a zero curl.

Electric potential

Electric potential at any point is defined as the amount of work done in bringing a unit positive charge from infinity to that point against the electrostatic force. It is represented by V.

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[1]From the fig $\hat{r} \cdot dl = dl\cos\theta$. In right angled triangle PQR (RQ is drawn normal to PQ) $\frac{dr}{dl} = \cos\theta$ or $dr = dl\cos\theta$.

As we know that the amount of work done in moving a charge between two points against the electrostatic force is equal to the line integral of electric field strength taken along the path joining the two points. So if a charge is moved from infinity to a point a, mathematically we can write electric potential as,

$$V = -\int_{\infty}^{a} E \cdot dl \tag{13}$$

If we have two points a and b, then electric potentials at these points are

$$V_a = -\int_{\infty}^{a} E \cdot dl$$

And

$$V_b = -\int_{\infty}^{b} E \cdot dl$$

Now the difference in potentials at points a and b is

$$V_b - V_a = -\int_{\infty}^{b} E \cdot dl + \int_{\infty}^{a} E \cdot dl = -\int_{\infty}^{b} E \cdot dl - \int_{a}^{\infty} E \cdot dl$$

$$= -\left(\int_{\infty}^{b} E \cdot dl + \int_{a}^{\infty} E \cdot dl\right)$$

$$V_b - V_a = -\int_{a}^{b} E \cdot dl = W_{ab}$$

This difference is called the *potential difference.* $\mathrm{W_{ab}}$is the work done in moving the unit positive charge between points a and b. *So the potential difference between any two points a and b is defined as the amount of work done in moving a unit positive charge between points a and b against the electrostatic force.*

In the previous section we have calculated the value for line integral of E between two points a and b, which is

$$\int_{a}^{b} E \cdot dl = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r_a} - \frac{1}{r_b}\right)$$

Where r_a and r_b are the distances of point a and b from the origin. If point a is ∞, we can define potential at point b as

$$V_b = -\int_{\infty}^{b} E \cdot dl = -\frac{q}{4\pi\epsilon_0}\left(\frac{1}{\infty} - \frac{1}{r_b}\right) = \frac{q}{4\pi\epsilon_0}\frac{1}{r_b}$$

This is the expression for electric potential at any point b. This expression holds for every point in space so writing V for V_b and r for r_b we can write the expression for the electric potential as

$$V = \frac{q}{4\pi\epsilon_0 r} \tag{14}$$

Electric field as negative gradient of electric potential

We proved earlier that electric field is a conservative field which means $\nabla \times E = 0$. The theorem on vector calculus says that if curl of a vector field is zero, the field can be expressed as gradient of a scalar potential. This theorem also holds for electric field and it can be expressed as negative gradient of electric potential. Let's prove it.

Electric potential at any point (say P) due a charge q at origin is

$$V = \frac{q}{4\pi\epsilon_0 r}$$

Taking the gradient of V, we have

$$\nabla V = \nabla\left[\frac{q}{4\pi\epsilon_0 r}\right] = \frac{q}{4\pi\epsilon_0}\nabla\left(\frac{1}{r}\right) = \frac{q}{4\pi\epsilon_0}\left(-\frac{\hat{r}}{r^2}\right) = -\frac{q}{4\pi\epsilon_0}\frac{\hat{r}}{r^2}$$

But $\frac{q}{4\pi\epsilon_0}\frac{\hat{r}}{r^2} = E$, the electric field due to charge q at point P.

Hence we have,

$$E = -\nabla V \tag{15}$$

This expression is very useful in calculating vector function E from a scalar function V.

Poisson's and Laplace's equations

We can write the fundamental equation of E (the Gauss law) in terms of V. The Gauss law is

$$\nabla \cdot E = \frac{\rho}{\epsilon_0}$$

As

$$E = -\nabla V$$

We can write

$$\nabla \cdot E = \nabla \cdot (-\nabla V) = -(\nabla \cdot \nabla)V = -\nabla^2 V$$

So we see that the divergence of E is equal to negative of Laplacian of V. Rewriting Gauss law, we have

$$\nabla^2 V = -\frac{\rho}{\epsilon_0} \tag{16}$$

This is known as **Poisson's equation.**

In the regions where there is no charge, we have $\rho = 0$ and then the Poisson's equation reduces to

$$\nabla^2 V = 0 \tag{17}$$

This is known as **Laplace's equation.**

The Poisson's and Laplace's equations are used to find electric potential in the regions having finite charge density and zero charge density respectively.

Electric quadrapole

We are familiar with electric monopole and electric dipole. A single charge is called a monopole. Two charges of equal magnitude and opposite sign separated by a small distance is called a dipole. A dipole has a finite dipole moment which in magnitude is the product of any one of the charge and separation between the two charges and has a direction from the negative charge to the positive charge. When two dipoles are arranged in such a way that that their dipole moments have the same magnitude but point in opposite directions they constitute a **quadrapole.** Thus an electric quadrapole consist of a charge distribution which is same as a special arrangement of two electric dipoles. The arrangement of two dipoles forming a quadrapole is shown in the figure (3a) below.

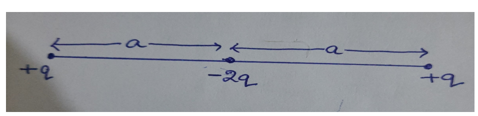

Figure 3a

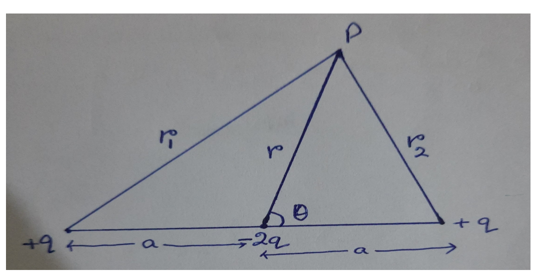

Figure 3b

Electric potential due to a quadrapole

Figure (3b) above shows a quadrapole of length $2a$ and P is some point at a distance r from the centre of the quadrapole where we wish to find the electric potential. Electric potential obeys superposition principle and hence its value at point P due to the quadrapole will be equal to the sum of electric potentials due to various point charges of the quadrapole.

Hence electric potential at point p is

$$V = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{r_1} - \frac{2q}{r} + \frac{q}{r_2}\right] \tag{18}$$

$$V = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{r_1} - \frac{2}{r} + \frac{1}{r_2}\right] \tag{19}$$

Now $r_1 = |\vec{r_1}| = |\vec{r} + \vec{a}| = (r^2 + a^2 + 2\vec{r}\cdot\vec{a})^{1/2} = (r^2 + a^2 + 2ra\cos\theta)^{1/2}$

Where θ is the angle between $\vec{r}$ and $\vec{a}$.

$$r_1 = r\left[1 + \frac{a^2}{r^2} + \frac{2a\cos\theta}{r}\right]^{1/2}$$

$$\frac{1}{r_1} = \frac{1}{r}\left[1 + \left(\frac{a^2}{r^2} + \frac{2a\cos\theta}{r}\right)\right]^{-1/2}$$

Expanding by Binomial theorem as $(1+x)^n = 1 + \frac{nx}{1!} + \frac{n(n-1)x^2}{2!} + \cdots$, we have

$$\frac{1}{r_1} = \frac{1}{r}\left[1 - \frac{1}{2}\left(\frac{a^2}{r^2} + \frac{2a\cos\theta}{r}\right) + \frac{3}{8}\left(\frac{a^2}{r^2} + \frac{2a\cos\theta}{r}\right)^2 - \cdots\right]$$

Since $r \gg a$, neglecting higher power terms (keeping terms only which contain second power of $\frac{a}{r}$), we have

$$\frac{1}{r_1} = \frac{1}{r}\left[1 - \frac{a^2}{2r^2} - \frac{a\cos\theta}{r} + \frac{3}{8}\cdot\frac{4a^2}{r^2}\cos^2\theta\right]$$

$$\frac{1}{r_1} = \frac{1}{r}\left[1 + \frac{a^2}{2r^2}(3\cos^2\theta - 1) - \frac{a\cos\theta}{r}\right] \tag{20}$$

Using the same logic we can find $\frac{1}{r_2}$, which is

$$\frac{1}{r_2} = \frac{1}{r}\left[1 + \frac{a^2}{2r^2}(3\cos^2\theta - 1) + \frac{a\cos\theta}{r}\right] \tag{21}$$

Substituting the values of equation (20) and (21) in equation (19), we get

$$V = \frac{q}{4\pi\epsilon_0 r}\left[1 + \frac{a^2}{2r^2}(3\cos^2\theta - 1) - \frac{a\cos\theta}{r} - 2 + 1 + \frac{a^2}{2r^2}(3\cos^2\theta - 1) - \frac{a\cos\theta}{r}\right]$$

Or

$$V = \frac{qa^2}{4\pi\epsilon_0 r^3}\left(3\cos^2\theta - 1\right) \tag{22}$$

This can also be written as follows.

$$V = \frac{Q_d}{8\pi\epsilon_0 r^3}\left(3\cos^2\theta - 1\right) \tag{23}$$

The product $2qa^2$ is called the electric quadrapole moment and is denoted by Q_d. Equation (23) gives the expression for electric potential due to a linear quadrapole which varies inversely as the cube of distance.

Electric field due to a quadrapole

As we have calculated the expression for electric potential V due to a quadrapole. Now using the relation $E = -\nabla V$ we can find electric field due to a quadrapole.

In component form, we have

$$E_x = -\frac{\partial V}{\partial x}, \qquad E_y = -\frac{\partial V}{\partial y}, \qquad E_z = -\frac{\partial V}{\partial z}$$

Since

$$V = \frac{qa^2}{4\pi\epsilon_0 r^3}\left(3\cos^2\theta - 1\right)$$

$$E_x = -\frac{\partial V}{\partial x} = -\frac{qa^2}{4\pi\epsilon_0}\frac{\partial}{\partial x}\left(\frac{3\cos^2\theta}{r^3} - \frac{1}{r^3}\right)$$

Using the relation $x = r\cos\theta$ or $\cos\theta = \frac{x}{r}$ between polar and Cartesian coordinates we have

$$E_x = -\frac{qa^2}{4\pi\epsilon_0}\frac{\partial}{\partial x}\left(\frac{3x^2}{r^5} - \frac{1}{r^3}\right) = -\frac{qa^2}{4\pi\epsilon_0}\left[\frac{6x}{r^5} + 3x^2\frac{(-5)}{r^6}\frac{\partial r}{\partial x} - \frac{(-3)}{r^4}\frac{\partial r}{\partial x}\right]$$

Since

$$r = (x^2 + y^2 + z^2)^{1/2}, \qquad \frac{\partial r}{\partial x} = \frac{x}{(x^2 + y^2 + z^2)^{1/2}} = \frac{x}{r}, \& \frac{\partial r}{\partial y} = \frac{y}{r} \tag{24}$$

Therefore

$$E_x = -\frac{qa^2}{4\pi\epsilon_0}\left[\frac{6x}{r^5} - \frac{15x^3}{r^7} + \frac{3x}{r^5}\right]$$

$$= \frac{3qa^2}{4\pi\epsilon_0 r^4}\left[\frac{5x^3}{r^3} - \frac{3x}{r}\right]$$

$$E_x = \frac{3qa^2}{4\pi\epsilon_0 r^4}(5\cos^3\theta - 3\cos\theta) \tag{25}$$

Also

$$E_y = -\frac{\partial V}{\partial y} = -\frac{qa^2}{4\pi\epsilon_0}\frac{\partial}{\partial y}\left(\frac{3x^2}{r^5} - \frac{1}{r^3}\right)$$

$$= -\frac{qa^2}{4\pi\epsilon_0}\left[3x^2\frac{(-5)}{r^6}\frac{\partial r}{\partial y} - \frac{(-3)}{r^4}\frac{\partial r}{\partial y}\right]$$

$$= \frac{qa^2}{4\pi\epsilon_0}\left[\frac{15x^2y}{r^7} - \frac{3y}{r^4}\right]$$

$$E_y = \frac{3qa^2}{4\pi\epsilon_0 r^5}\left[\frac{5x^2}{r^2} - 1\right]y = \frac{3qa^2}{4\pi\epsilon_0 r^5}(5\cos^2\theta - 1)y \tag{26}$$

Similarly

$$E_z = \frac{3qa^2}{4\pi\epsilon_0 r^5}\left[\frac{5x^2}{r^2} - 1\right]z = \frac{3qa^2}{4\pi\epsilon_0 r^5}(5\cos^2\theta - 1)z \tag{27}$$

After calculating the expressions for each component of E we can find the magnitude of E using the expression given below

$$E = \left(E_y^2 + E_y^2 + E_y^2\right)^{\frac{1}{2}}$$

$$E = \frac{3qa^2}{4\pi\epsilon_0}\left[\frac{1}{r^8}(5\cos^3\theta - 3\cos\theta)^2 + \frac{1}{r^{10}}(5\cos^2\theta - 1)^2y^2 + \frac{1}{r^{10}}(5\cos^2\theta - 1)^2z^2\right]^{1/2}$$

$$= \frac{3qa^2}{4\pi\epsilon_0 r^4}\left[(5\cos^3\theta - 3\cos\theta)^2 + \frac{1}{r^2}(5\cos^2\theta - 1)^2(y^2 + z^2)^2\right]^{1/2}$$

Since

$$y^2 + z^2 = r^2 - x^2 = r^2 - r^2\cos^2\theta = r^2(1 - \cos^2\theta)$$

$$E = \frac{3qa^2}{4\pi\epsilon_0 r^4}[(5\cos^3\theta - 3\cos\theta)^2 + (5\cos^2\theta - 1)^2(1 - \cos^2\theta)]^{1/2}$$

On simplifying the above expression, we get

$$E = \frac{3qa^2}{4\pi\epsilon_0 r^4}\left[5\cos^4\theta - 2\cos^2\theta + 1\right]^{1/2} \tag{28}$$

This expression gives the electric field intensity at any point due to a quadrapole.

Energy of electrostatic field

We have already studied that work done in moving a charge Q between two points a and b against the electric force $F = QE$ (the force one must exert to move the charge in opposition to this electric force is $F = -QE$) is given as

$$W_{ab} = \int_a^b F \cdot dl = -Q \int_a^b E \cdot dl$$

Also the electric potential difference between two points a and b is given by

$$V_b - V_a = \int_a^b E \cdot dl$$

So we can write work done as

$$W_{ab} = Q(V_b - V_a)$$

In particular if you want to bring the charge from infinity to point r, the work done is

$$W = Q(V - V_\infty) = QV \tag{29}$$

Where V is the potential at point r. Electric potential at infinity is zero. Thus we see that work done can be taken as product of charge Q and potential V.

The practical example of transferring charge is in case of charging a capacitor. So consider a parallel plate capacitor in which charge is transferred to its plates by the battery. In doing so work is done and this work done is stored in the capacitor in the form of electrostatic energy. If q is the charge on the plates of the capacitor and V is the potential difference between the plates then $q = CV$, where C is called the capacitance of the capacitor. Now small amount of work done in supplying charge dq is

$$dW = Vdq = \frac{q}{C}dq$$

Total work done in charging the capacitor from 0 to q is

$$W = \int dW = \int \frac{q}{C}dq = \frac{q^2}{2C}$$

This work done is stored in the capacitor in the form of electrostatic potential energy ($\mathcal{E}$).
Thus we write energy stored as

$$\mathcal{E} = \frac{q^2}{2C} = \frac{1}{2}CV^2 \tag{30}$$

If A is the area of each plate of the capacitor and t is the distance between the plates then

$$C = \frac{\epsilon_0 A}{t} \tag{31}$$

And the potential difference between the plates is given by

$$V = Et \tag{32}$$

Putting these two values from equation (31) and (32) in equation (30), we get

$$\mathcal{E} = \frac{1}{2}\epsilon_0 E^2 At$$

And hence the energy density i.e. energy per unit volume is given by

$$\varepsilon = \frac{\mathcal{E}}{At} = \frac{1}{2}\epsilon_0 E^2 \tag{33}$$

Dielectrics

In this chapter we studied the electric fields in free space. In sections to follow we will study the electric fields in matter. Matter is broadly classified as **conductors** and **insulators** (also called **dielectrics**). Conductors are those materials which contain enormous amount of free charges in it. These charges can move through the conductors randomly and when placed in electric field they move in specific directions depending upon the direction of electric fields. By free charges we mean that all the electrons (charges) are not bound to the atomic nucleus, but some of them move randomly. In dielectrics there are no free charges because the electrons are tightly bound to the atomic nucleus. They can make small displacements but only within the atom or molecule. These small movements are not such that they can resemble the arrangements of charges in the conductor, but their influence is accountable for characteristics behaviour of dielectric materials. Dielectrics being insulators are expected to be not affected by external electric fields but when placed in external fields they show certain variations. (You must have studied that when a dielectric is placed between the plates of a parallel plate capacitor its capacitance is increased by a factor K known as the dielectric constant). We will study the effect of external electric field on dielectrics in the following sections.

Polar and non-polar molecules

Depending upon the distribution of charges within dielectrics its molecules can be classified as polar and non-polar. If the charge distribution is such that the centre of gravity of positive and negative charge coincides exactly we call such molecules as **non-polar molecules**. This means that there is no separation between the positive charge centre and the negative charge centre and hence we can say that non-polar molecules have zero net dipole moment. Non-polar molecules, therefore, do not have any permanent dipole moment. A non-polar molecule has a symmetrical shape. Examples of non-polar molecules include H_2, O_2, N_2, CO_2 and CH_4. On the other hand, if the charge distribution is such that the centre of gravity of positive charge does not coincide with the centre of gravity of negative charge we call such molecules as **polar molecules**. Such molecules, therefore, constitutes permanent dipoles and hence have non-zero net dipole moment. Polar molecules are not symmetrical in shape. HCl, NH_3, H_2O and N_2O are examples of polar molecules.

Polarisation of dieletric: Induced dipoles

To know the overall effect of external electric field on dielectric we will start with one atom of the dielectric. We all know that an atom is electrically neutral (i.e. it has no charge). If it is placed in an external electric field our guess is that there will be no effect on the atom. But this is not correct. This is because if we look at the internal structure of the atom it consists of positively nucleus with negatively charged electrons revolving around it. These two regions of charge within the atom are affected by the external field. The nucleus is pushed in the direction of electric field (electric field is directed from positive end to the negative end) whereas the negative charge cloud is pushed in the opposite direction. The atom now behaves as a tiny dipole with positive and negative charges moved slightly apart. We call them **induced dipoles** and the atom is said to be **polarised**. The atom now has tiny dipole moment **p**, whose direction is same as that of the external electric field. See figure 4a.

Thus **polarisation** is the process in which the centre of positive charge and centre of negative charge of the molecules of a dielectric are separated inducing tiny dipoles when placed in external electric field.

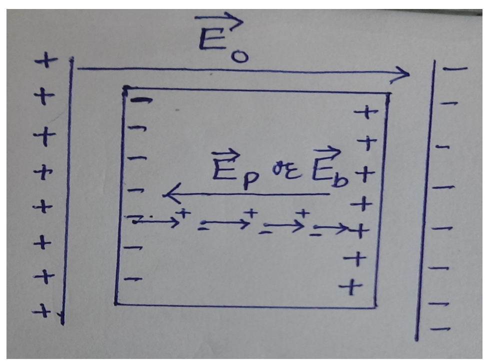

Figure 4a

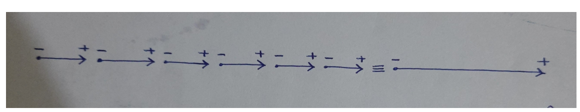

Figure 4b

Note that the force pulling apart the nucleus and the electrons due to external electric field is balanced by the force of attraction drawing them together. This leaves the atoms of the dielectric polarised. If the force due to external field is strong enough it will pull apart the atom completely and ionize the atom.

When a dielectric is placed in external electric field it gets polarised. Tiny dipoles corresponding to each atom are induced in it. These dipoles align themselves according to the external electric field throughout the dielectric as shown in the fig. If we look across any length of the dielectric we will see a long string of dipoles from one end to the other (see fig 4b). Along the line the positive charge of one dipole cancels the negative charge of the other which is adjacent to it, leaving two equal and opposite charges at the ends. This goes out the same for every length of the dielectric creating one end of the dielectric negatively charged and the other end positively charged. We call the net charge at the ends **bound charges.** The word bound is used to remind us that these charges cannot be removed. The bound charges are the result of polarisation and we can also call them polarisation charges.

Polar molecules where there are permanent dipoles also show polarisation. In this case when external electric field is applied the permanent dipoles (which have random orientation in absence of external electric field) experience a torque and align along the direction of electric field. By the same logic as above bound charges are induced on the ends of the dielectric and the dielectric is said to have acquired polarisation.

Due to polarisation and electric field is set up in the dielectric whose direction is opposite to that of external electric field. If E_0 is the external applied field and E_b is the electric field due to bound charges then the resultant electric field is given by

$$E = E_0 - E_b$$

Thus the bound charges reduce the value of electric field in the dielectric.

The induced dipole moment is directly proportional to the external electric field (as long as the field is not too strong)

$$p = \alpha E_0 \tag{34}$$

The constant of proportionality α is called **atomic polarizability**. Polarizabilty is the ability of the molecules to acquire a dipole moment p in response to external applied field E_0. Its value depends on the detailed structure of the atom.

Polarisation vector (P)

Polarisation vector is the measure of degree of polarisation in a dielectric. It is defined as the dipole moment per unit volume of the dielectric and is denoted by $\boldsymbol{P}$. If we have a dielectric slab of area A and thickness t, then polarisation vector is

$$P = \frac{dipole\ moment}{volume} = \frac{q_b t}{At} = \frac{q_b}{A}$$

Here q_b is the bound charge on the dielectric slab. Also charge per unit area is the surface charge density, therefore, $\frac{q_b}{A} = \sigma_b$ is the surface charge density of bound charges. Therefore we have

$$P = \frac{q_b}{A} = \sigma_b \tag{35}$$

Displacement vector (D)

Consider a parallel plate capacitor carrying a charge $-q$ and $+q$ on the plates establishing an electric field E_0. See fig 5.

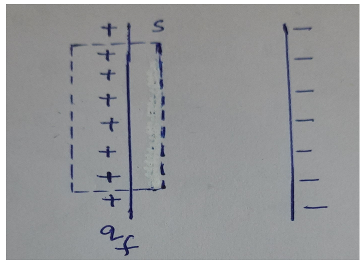

Figure 5

Now the Gauss law for the Gaussian surface S shown by the dotted lines reads

$$\oiint E_0 \cdot da = \frac{q}{\epsilon_0}$$

$$E_0 \oiint da = \frac{q}{\epsilon_0}$$

If A is the area of the Gaussian surface, we have

$$E_0 A = \frac{q}{\epsilon_0}$$

$$E_0 = \frac{q}{\epsilon_0 A} \tag{36}$$

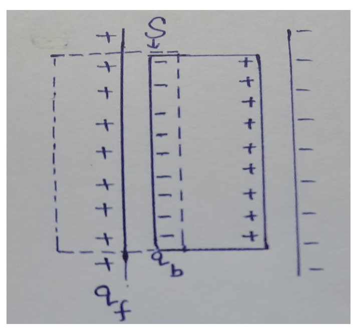

Figure 6

Now a dielectric slab introduced between plates of a parallel plate capacitor (fig 6). This electric field E_0 acts as external electric field for the dielectric and polarises it. The polarisation field opposite to E_0 reduces the resultant field to E. Applying Gauss law to the Gaussian surface S, we have

$$\oiint E \cdot da = \frac{q - q_b}{\epsilon_0} \tag{37}$$

$$EA = \frac{q - q_b}{\epsilon_0} \tag{38}$$

$$\epsilon_0 E = \frac{q}{A} - \frac{q_b}{A}$$

But $\frac{q_b}{A} = P$, the polarisation vector

So we have

$$\epsilon_0 E = \frac{q}{A} - P$$

$$\epsilon_0 E + P = \frac{q}{A}$$

The term on the left hand side is called the electric displacement vector $\boldsymbol{D}$. So we have,

$$D = \epsilon_0 E + P \tag{39}$$

Electric displacement vector is defined as the charge on the plates of the capacitor per unit area. The charge q is charge other than the bound charge (here it is the charge on the plates of the capacitor). We call this charge as **free charge**, which can include electrons on the conductor, ions within the dielectrics or any other charge which is not the result of polarisation. Hence we can represent this charge as q_f.

We can, therefore, write

$$D = \frac{q_f}{A} = \epsilon\left(\frac{q_f}{\epsilon A}\right) = \epsilon\left(\frac{\sigma_f A}{\epsilon A}\right) = \epsilon\left(\frac{\sigma_f}{\epsilon}\right) = \epsilon E$$

Here we have used $q_f = \sigma_f A$ and $\frac{\sigma_f}{\epsilon} = E$. So we have

$$D = \epsilon E \tag{40}$$

Thus D is proportional to E. Here ϵ is called the **permittivity** of the material. The ratio of permittivity of the material to the permittivity of free space is **relative permittivity** or **dielectric constant** denoted by K or ϵ_r. That is

$$\epsilon_r \text{ or } K = \frac{\epsilon}{\epsilon_0}$$

Gauss law in presence of dielectric

Gauss law for a dielectric placed between a parallel plate capacitor is written in equation (38) as (replacing q_f for q)

$$\oiint E \cdot da = \frac{q_f - q_b}{\epsilon_0} \tag{41}$$

$$EA = \frac{q_f - q_b}{\epsilon_0}$$

$$E = \frac{q_f - q_b}{\epsilon_0 A} \tag{42}$$

Now the dielectric constant K is the ratio of applied electric field to the reduced value of electric field

$$K = \frac{E_0}{E}$$

Using equations (36) (replacing q_f for q) and (41) we can write

$$K = \frac{q_f}{q_f - q_b}$$

$$q_f - q_b = \frac{q_f}{K} \tag{43}$$

Writing equation (41) using (43) we get

$$\oiint E \cdot da = \frac{q_f}{\epsilon_0 K} = \frac{q_f}{\epsilon}$$

$$\oiint \epsilon E \cdot da = q_f$$

Using equation (40), we have

$$\oiint D \cdot da = q_f \tag{44}$$

This is the Gauss law for dielectrics in **integral form**.

If ρ_f is the volume charge density of free charges then the total charge within the volume $d\tau$ of the dielectrics we have

$$q_f = \iiint \rho_f d\tau \tag{45}$$

Using Gauss divergence theorem we can write

$$\oiint D \cdot da = \iiint (\nabla \cdot D)\, d\tau \tag{46}$$

Then we can use equations (45) and (46) to rewrite equation (44) as

$$\iiint (\nabla \cdot D)\, d\tau = \iiint \rho_f d\tau$$

$$\iiint \left(\nabla \cdot D - \rho_f\right) d\tau = 0$$

Since this equation is true for every volume, the integrant must be equal to zero.

$$\nabla \cdot D - \rho_f = 0$$

$$\nabla \cdot D = \rho_f \tag{47}$$

This is Gauss law in presence of dielectrics in **differential form**.

This is very useful way to express Gauss law because it makes reference to free charges only and free charges are what we can deal with or control. In any problem we know about free charges but we initially do not know about bound charges. Bound charges are produced only when we put free charges in place.

Divergence of P

We know that the field due to the polarisation is the field due to the bound charges. Within the dielectrics the total charge density (ρ) is equal to the sum of bound charge density (ρ_b) and free charge density (ρ_f). Thus,

$$\rho = \rho_b + \rho_f$$

The Gauss law in differential form reads

$$\nabla \cdot E = \frac{\rho}{\epsilon_0} = \frac{\rho_b + \rho_f}{\epsilon_0}$$

$$\nabla \cdot \epsilon_0 E = \rho_b + \rho_f \tag{48}$$

From equation (36) we can write

$$\epsilon_0 E = D - P$$

Using this equation we can write equation (48) as

$$\nabla \cdot (D - P) = \rho_b + \rho_f$$

$$\nabla \cdot D - \nabla \cdot P = \rho_b + \rho_f$$

Using equation (47) ,we have, the divergence of P as

$$\nabla \cdot P = -\rho_b \tag{49}$$

Electric susceptibility

Polarisation of a dielectric is due to the application of electric field to the dielectric. When external field is applied it polarises the dielectric and this polarisation will produce its own field which then results into the net field, and this in turns modifies polarisation. So the polarisation is the result of the net field E, and thus the polarisation vector P is proportional to the net field.

$$P \propto E$$

$$P = \epsilon_0 \chi_e E \tag{50}$$

The constant of proportionality χ_e is called **electric susceptibility**. The factor ϵ_0 is introduced to make χ_e dimensionless. Materials which obey equation (50) are called linear dielectrics. Electric susceptibility is the measure of the ease with which the dielectric medium can be polarised.

The displacement vector D is given by

$$D = \epsilon_0 E + P = \epsilon_0 E + \epsilon_0 \chi_e E = \epsilon_0 (1 + \chi_e) E$$

$$D = \epsilon_0 (1 + \chi_e) E$$

Comparing this equation with equation (40) we get

$$\epsilon = \epsilon_0 (1 + \chi_e) \tag{51}$$

In vacuum or free space where there is no material to polarise the electric susceptibility is zero and the permittivity $\epsilon = \epsilon_0$, (called the permittivity of vacuum or free space). Now from equation (51) we can write

$$\frac{\epsilon}{\epsilon_0} = 1 + \chi_e$$

The left hand side of this equation is called relative permittivity or dielectric constant of the material which is denoted by K or ϵ_r. Thus we have

$$K = 1 + \chi_e \tag{52}$$

This gives the relationship between dielectric constant and electric susceptibility.

Energy in the Dielectric System

Energy stored in a dielectric is the amount of work done in moving the free charges and put them all in their respective positions. We moved the free charges only because that is what we can move, bound charges are fixed. Therefore work done in arranging the free charge over a volume $d\tau$ is

$$W = \frac{1}{2} \iiint \rho_f V d\tau \tag{53}$$

Where V is the electric potential.

According to the Gauss law for dielectric we have

$$\nabla \cdot D = \rho_f$$

So equation (53) becomes

$$W = \frac{1}{2} \iiint V (\nabla \cdot D) d\tau$$

But using $\nabla \cdot (DV) = D \cdot (\nabla V) + V(\nabla \cdot D) \Rightarrow V(\nabla \cdot D) = \nabla \cdot (DV) - D \cdot (\nabla V)$

$\Rightarrow V(\nabla \cdot D) = \nabla \cdot (DV) + D \cdot E \qquad [\because E = -\nabla V]$

Therefore

$$W = \frac{1}{2}\iiint [\nabla \cdot (DV) + D \cdot E]d\tau$$

$$= \frac{1}{2}\iiint \nabla \cdot (DV)d\tau + \frac{1}{2}\iiint (D \cdot E)d\tau$$

Using the Gauss divergence theorem, we have

$$\iiint \nabla \cdot (DV)d\tau = \oiint DV \cdot da$$

Therefore

$$W = \frac{1}{2}\oiint DV \cdot da + \frac{1}{2}\iiint (D \cdot E)d\tau$$

If we integrate over the whole surface the surface S is taken as infinity, in that case the potential at infinity becomes zero. Therefore

$$\oiint DV \cdot da = 0$$

So

$$W = \frac{1}{2}\iiint (D \cdot E)d\tau \tag{54}$$

This work done is stored as the electrostatic energy $\mathcal{E}$ of the dielectrics.

$$\mathcal{E} = \frac{1}{2}\iiint (D \cdot E)d\tau$$

This expression can be written in another form using the expression $D = \epsilon E$ as

$$\mathcal{E} = \frac{1}{2}\iiint (\epsilon E \cdot E)d\tau = \frac{1}{2}\iiint \epsilon E^2 d\tau$$

Boundary conditions satisfied by *E* and *D* at the interface between two homogenous dielectrics

When the fields cross the boundary of two homogenous dielectrics they show a discontinuity. We will check for this discontinuity here. For this we will make use of two fundamental equations. (1) Closed line integral of electric field,

$$\oint E \cdot dl = 0 \tag{55}$$

And (2) Gauss law in dielectric medium,

$$\oiint D \cdot da = q_f \tag{56}$$

Consider two homogenous dielectrics 1 and 2 having permittivity ϵ_1 and ϵ_2 separated by a boundary C. Let E_1 and E_2 be the electric field intensities in these media. The electric fields have two components each as shown in figure (7), one is called the normal component and other is called the transverse component. Consider a closed loop ABCD, the closed line integral of electric field over which can be written as

$$\oint E \cdot dl = \int_A^B E \cdot dl + \int_B^C E \cdot dl + \int_C^D E \cdot dl + \int_D^A E \cdot dl$$

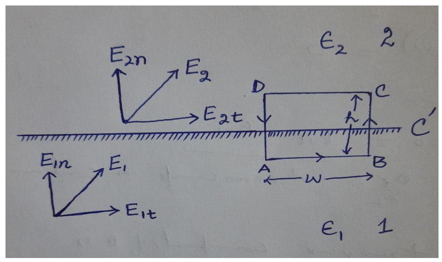

Figure 7

Along the lengths AB and CD, the normal components E_{1n} and E_{2n} contribute nothing to the line integral because $E_{1n} \cdot dl = E_{1n} dl \cos 90° = 0$ and $E_{2n} \cdot dl = E_{2n} dl \cos 90° = 0$. With the same arguments, along the lengths BC and DA, the transverse components E_{1t} and E_{2t} make no contribution to the line integral. Therefore the closed line integral can be written as

$$\oint E \cdot dl = (E_{1t} \cdot w)_{AB} + \left(E_{1n} \cdot \frac{h}{2} + E_{2n} \cdot \frac{h}{2}\right)_{BC} + (E_{2t} \cdot w)_{CD} + \left(E_{2n} \cdot \frac{h}{2} + E_{1n} \cdot \frac{h}{2}\right)_{DA}$$

$$= E_{1t} w + E_{1n} \frac{h}{2} + E_{2n} \frac{h}{2} - E_{2t} w - E_{2n} \frac{h}{2} - E_{1n} \frac{h}{2}$$

$$= E_{1t} w - E_{2t} w$$

Now using equation (55), we get

$$E_{1t} w - E_{2t} w = 0 \quad \Rightarrow (E_{1t} - E_{2t}) w = 0 \quad \Rightarrow E_{1t} - E_{2t} = 0$$

$$E_{1t} = E_{2t} \tag{57}$$

Thus the transverse component of electric field is continuous across the boundary.

Now as $D = \epsilon E$ or $E = \frac{D}{\epsilon}$ therefore from equation (57) we have

$$\frac{D_{1t}}{\epsilon_1} = \frac{D_{2t}}{\epsilon_2} \tag{58}$$

This shows that the transverse component of D is discontinuous across the bonndary.

Now if D_1 and D_2 are the displacement vectors in the two media with their normal components D_{1n} and D_{2n} and transverse components D_{1t} and D_{2t} as shown (fig 8). Consider a small pill box shaped surface across the two media so that closed surface integral of D over this surface is

$$\oiint D \cdot da = \left(\iint D \cdot da\right)_{top} + \left(\iint D \cdot da\right)_{bottom} + \left(\iint D \cdot da\right)_{curved}$$

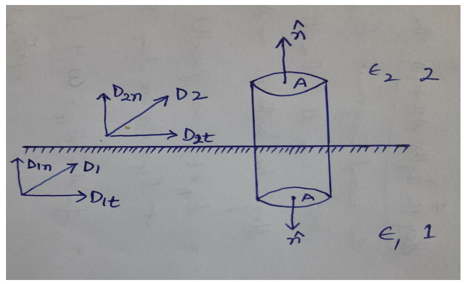

Figure 8

By symmetry the integral $(\iint D \cdot da)_{curved} = 0$. For the top and bottom surfaces the transverse components of D contribute nothing, and therefore, taking the normal components of D, we have

$$\oiint D \cdot da = D_{2n} \cdot A + D_{1n} \cdot A = D_{2n}A - D_{1n}A$$

Where A is the area of the top and bottom surfaces with equal magnitude and opposite direction. Using equation (56), we get

$$D_{2n}A - D_{1n}A = q_f$$

If σ_f is the surface charge density then the total surface charge is $q_f = \sigma_f A$. Then the above equation can be written as

$$D_{2n}A - D_{1n}A = \sigma_f A$$

$$D_{2n} - D_{1n} = \sigma_f \tag{59}$$

Thus the normal component of D changes by an amount equal to σ_f across the boundary of the two media. If we have a charge free surface then $\sigma_f = 0$, and then the normal component of D is continuous given by relation

$$D_{2n} - D_{1n} = 0$$

$$D_{1n} = D_{2n}$$

This expression can also be written as

$$\epsilon_1 E_{1n} = \epsilon_2 E_{2n}$$

The normal component of electric field is discontinuous across the boundary.